Lecture :
Some applications of z-transform

- Quick review of z-transform
- Rational z-transform: poles, zeros, system function
- Time-domain behavior of LT systems. Stability and causality
- Inverse z-transform using partial fraction expansion

Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) dz$$

- Relation to Laplace transform:

$$z = e^{s\Delta t}$$

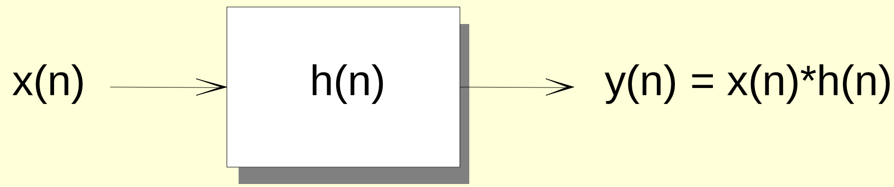
- Relation to discrete time Fourier transform:

$$z = 1 \cdot e^{j\omega}$$

Movie – evaluation of Fourier transform (demo1, demo2)

Rational Z-transform

- Z-transform used to characterize systems.
- Convolution property:



$$y(n) = x(n) * h(n) \Rightarrow Y(z) = X(z)H(z)$$

- H(z) – system function

Rational z-transform: System function

- LCCDE system: $$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

- Using linearity and time shift properties:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Downarrow$$

$$Y(z) = H(z)X(z)$$

$$\boxed{H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}}$$

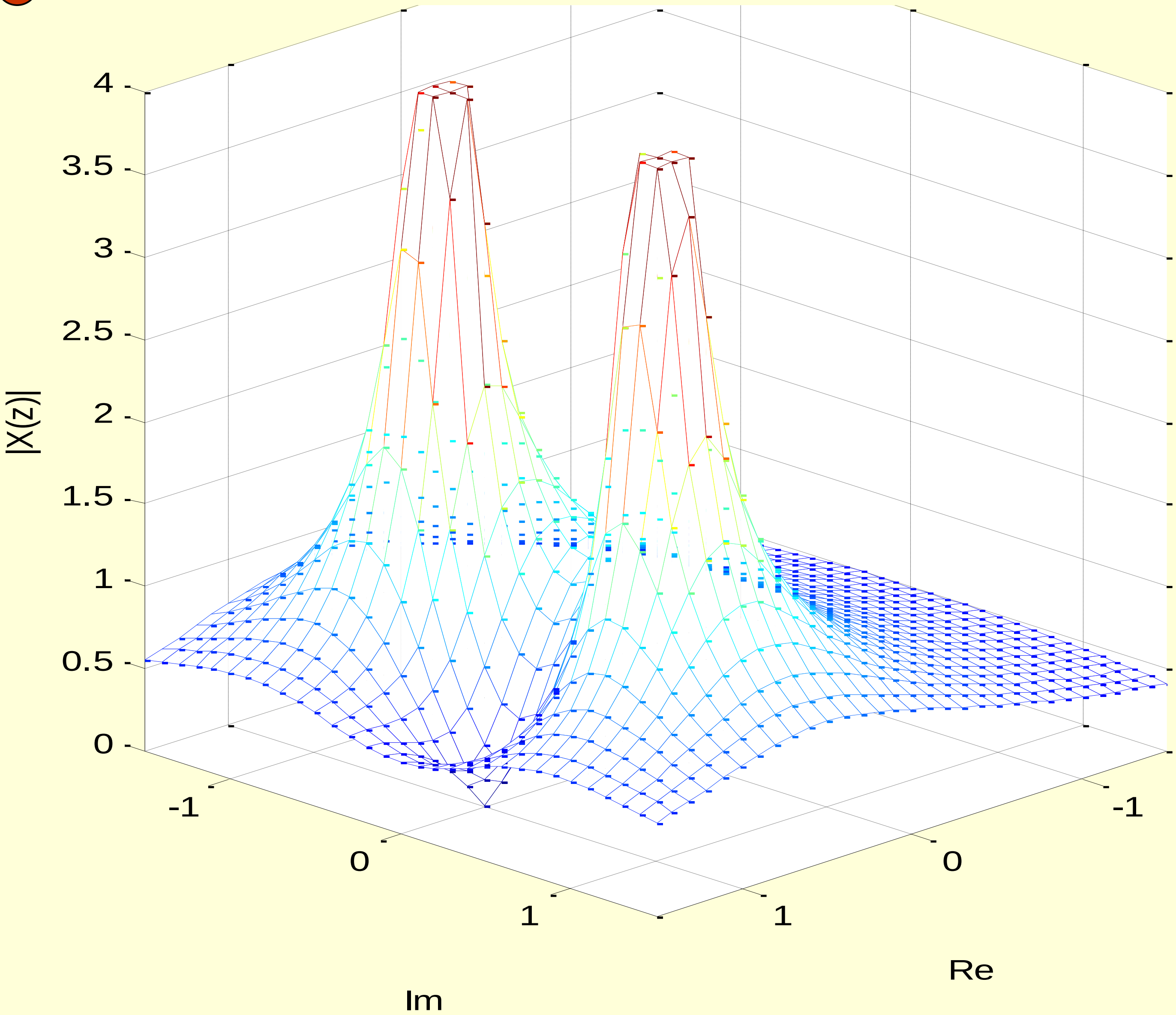
Poles and zeros

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$X(z) = \frac{b_0}{a_0} z^{N-M} \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

- The transform has M finite zeros at $z = z_1, \ldots z_M$, N infinite poles at $z=p_1 \ldots z_N$ and $|N-M|$ zeros (if $N>M$) or poles (if $N<M$) at the origin $z = 0$.

Pole-zero plots



Time-domain behavior of systems

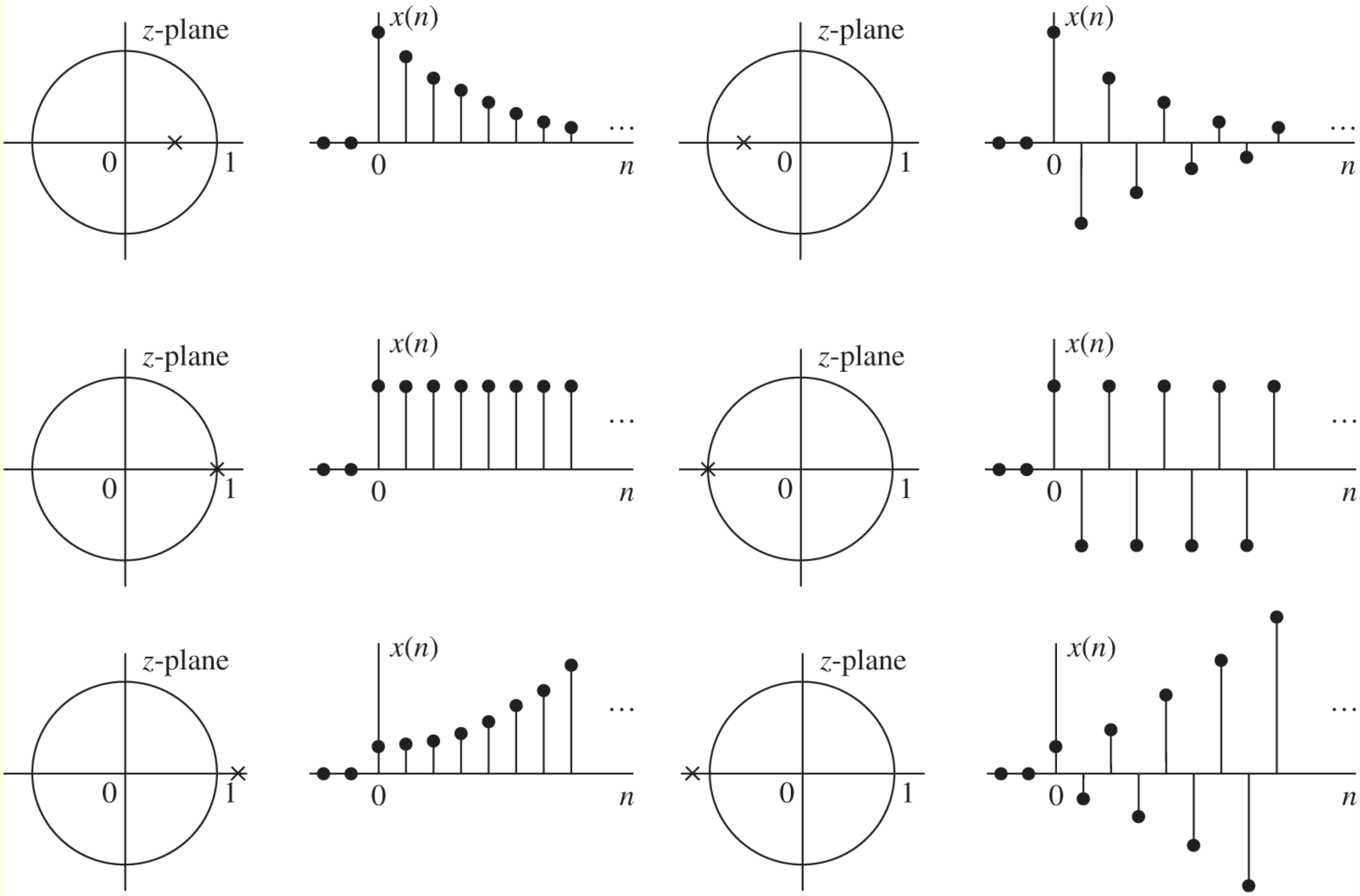


Figure 3.3.5 Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

Time domain behavior

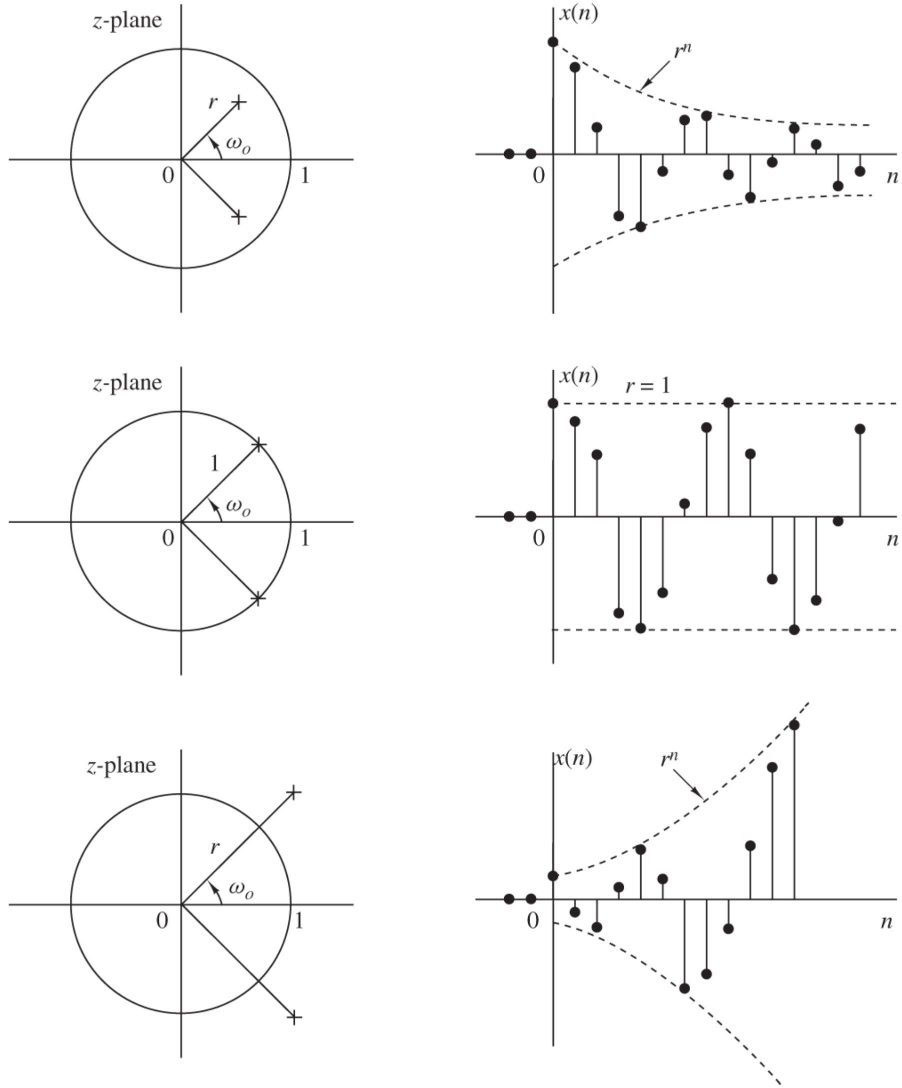


Figure 3.3.7 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

PEZ demo

Inverse Z-tr.: partial fraction expansion

- Distinct poles:

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

$$\boxed{A_k = \left.\frac{(z - p_k)X(z)}{z}\right|_{z=p_k}}$$

$$Z^{-1}\left\{\frac{1}{1 - p_k z^{-1}}\right\} = \begin{cases} (p_k)^n u(n), & \text{if ROC:}|z| > |p_k| \\ -(p_k)^n u(-n-1), & \text{if ROC:}|z| < |p_k| \end{cases}$$

Example with multiple poles

$$\boxed{X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}}$$

$$A_1 = \left.\frac{(z+1)X(z)}{z}\right|_{z=-1} = \frac{1}{4}$$

$$A_3 = \left.\frac{(z-1)^2 X(z)}{z}\right|_{z=1} = \frac{1}{2}$$

$$A_2 = \frac{d}{dz}\left[\frac{(z-1)^2 X(z)}{z}\right]_{z=1} = \frac{3}{4}$$

Stability

- A linear time-invariant system is BIBO stable if and only if the region of convergence of the system function includes the unit circle.
- A *causal* system is stable if all poles are inside the unit circle

Pole – zero cancellation

- Pole-zero cancellation occurs either in the system function itself, or in the product of the system function with the z-transform of input system.
- This is used to design filters.

Summary

- Z-transform useful in system analysis
- Can be used to determine Fourier transform
- Used to decide if a system is causal and stable
- Used to determine the response of a system
- Used to design filters through pole-zero cancellation.